Some Results on Dom Strong Domination Number and Maximum Degree of a Graph

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Abstract:

A subset S of V is said to be dom strong dominating set if for every vertex v $\in V - S$,there existsu$_1$,u$_2$ ∈ S such that u$_1$v , u$_2$ v ∈ E(G) and d(u$_1$) ≥ d(v).The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\boldsymbol{\gamma_{ds}}$ (G).The maximum degree of an undirected graph is the maximum of the degrees of its vertex and is denoted by $\Delta(G)$. In this paper the sum of the dom strong domination number and maximum degree of graph is obtained and characterized the corresponding extremal graphs.

Keywords: Dom strong domination number and Maximum degree.

1.Introduction

A graph is an ordered pair $G = (V, E)$ comprising a set V of vertices together with a set E of edges. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d\ (u)$. The vertex u and the edge x are incident with each other. If two distinct edges x and y are incident with the common vertex then they are called adjacent edges. The maximum degree of the graph is the maximum of the degrees of its vertex and is denoted by $\Delta(G)$.For terminology we refer toHarary[4]. A set $S \subseteq V$ is a dominating set of G if every vertex of G is dominated by some vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A subset S of V is called a dom strong dominating set if for every $v \in V - S$,there exists $u_1, u_2 \in S$ such that $u_1 v$, $u_2 v \in$ E(G) and $d(\ u_1) \geq d(\ v)$. The minimum cardinality of a dom strong dominating set is called the dom strong domination number and is denoted by $\gamma_{ds}(G)$.

Several authors have studied the problem of obtaining an upper bound for the sum of a dominating parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [7], the authors found an upper bound for the sum of the domination number and connectivity of graphs and characterized the corresponding extremal graphs. Motivated by the above,we find an upper bound for sum of the dom strong domination number and maximum degree of a graph and characterized the corresponding extremal graphs.

2.Preliminaries

Theorem 2.1:[5]For any connected graph G,$2 \leq \gamma_{ds} \leq n$

Theorem 2.2: [5]For any connected graph G,$\gamma_{ds}(G)$ = $n$if and only if G is a star.

Notation 2.3:$K_n(P_k)$ is the graph by attaching the end verticesof P_k path graph to any one vertex of the complete graph.$K_n(P_k, P_m, 0, \ldots)$ is the graph by attaching the end vertex of P_k path graph to any one of the vertex K_n and attaching the end vertex ofP_m path graph to another vertex of K_n.The 5-vertex tree is the graphby attaching one ofend vertex of $P_3$to another vertex of P_3 with degree 2,calledchair graph or fork graph or 'h'graph.$(K_4 - e)(nP_2)$ is a graph obtained by attaching one vertex of $P_2$with one vertex of $K_4 - e$ with maximum degree.

3. Relation between dom strong domination number and maximum degree.

Theorem 3.1: For any connected graph G, $\gamma_{ds}(G) + \Delta(G) \leq 2n - 1$and the equality holds if and only if G is a star.

Proof: If G is connected,$\gamma_{ds}(G) + \Delta(G) \leq n + n - 1 = 2n - 1$. Let$\gamma_{ds}(G) + \Delta(G) = 2n - 1$. Since$\gamma_{ds}(G) = n$,G must be a star whose maximum degree is n-1.Conversely, if G is a star then we can easily prove that $\gamma_{ds}(G) + \Delta(G) = 2n - 1$.

Theorem 3.2: For any connected graph G, $\gamma_{ds}(G) + \Delta(G) = 2n - 2$ if and only if$G$is isomorphic to K_3or $K_3(nP_2)$ where n is any positive integer.

Proof: If$G \cong K_3$or$K_3(nP_2)$then$\gamma_{ds}(G) + \Delta(G) = 2n - 2$.Suppose$\gamma_{ds}(G) + \Delta(G) = 2n - 2$ then the possible cases are (i) $\gamma_{ds}(G) = n$and $\Delta(G) = n - 2$ (ii) $\gamma_{ds}(G) = n - 1$and $\Delta(G) = n - 1$.
Case (i):$\gamma_{ds}(G) = n$ and $\Delta(G) = n - 2$.
Since$\gamma_{ds}(G) = n$, G must be a star. But maximum degree of a star is $n - 1$ which is a contradiction.
Case (ii):$\gamma_{ds}(G) = n - 1$ and$\Delta(G) = n - 1$.
Let $DS = \{u_1, u_2, \ldots u_{n-1}\}$ and$V - DS = \{v_1\}$.Suppose $< DS >$ is connected, if$|DS| = 1$ then condition fails. If $|DS| = 2$and if $d(u_1) = d(u_2) = n - 1$ then $G \cong K_3$.If$|DS| = 3$,$d(u_2) > d(u_i)$for$i = 1 or 3$and $d(u_2) = n - 1, d(u_3) = n - 2$ then$G \cong K_3(P_2)$. If $|DS| \geq 4$ and$< DS >= K_{1,n}$ then G is isomorphic to $K_3(nP_2)$.Forother cases, graph does not exist.

Theorem 3.3: For any connected graph G, $\gamma_{ds}(G) + \Delta(G) = 2n - 3$ if and only if$G$ is isomorphic toK_4or $K_4 - e$ or fork graph or P_4or $K_3(nP_2, P_2, 0)$, for n≥ 0or any of the graphs given in the figure 3.1.

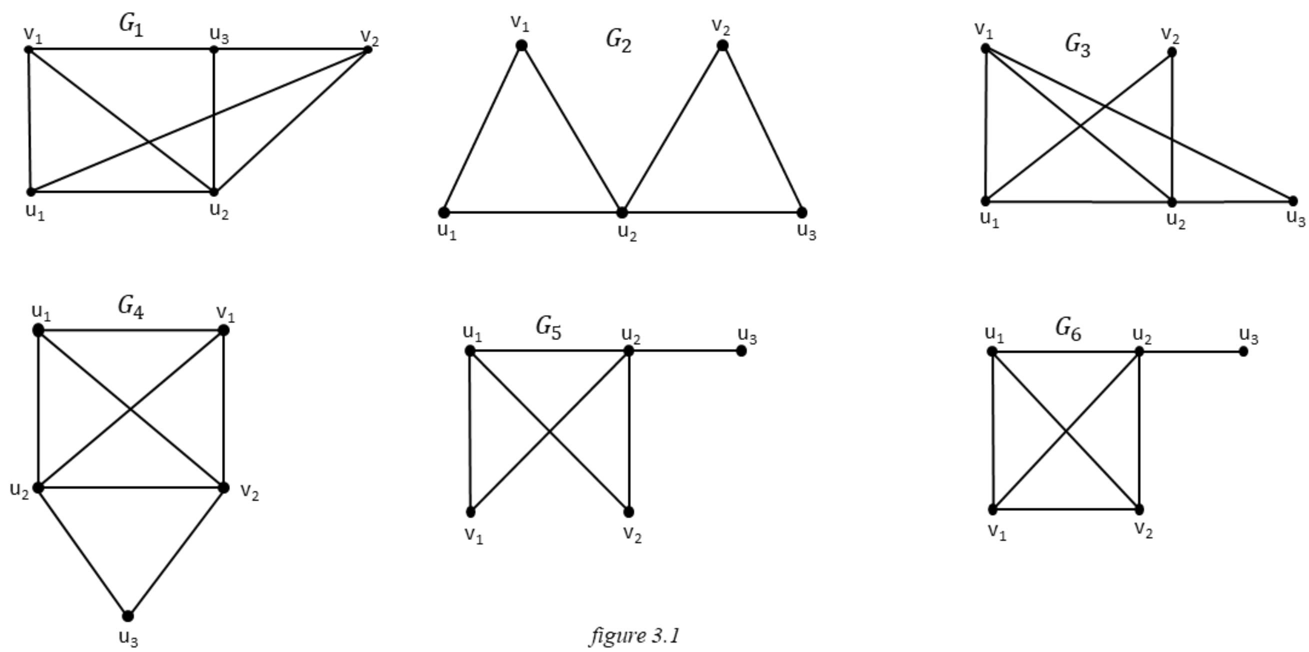

figure 3.1

Proof: If$G \cong K_4$or $K_4 - e$ or fork graph or P_4or $K_3(nP_2, P_2, 0)$ for n≥ 0 or any of the graphs given in the figure 3.1 then $\gamma_{ds}(G) + \Delta(G) = 2n - 3$.Suppose $\gamma_{ds}(G) + \Delta(G) = 2n - 3$then the possible cases are(i)$\gamma_{ds}(G) = n$and $\Delta(G) = n - 3$ (ii) $\gamma_{ds}(G) = n - 1$ and $\Delta(G) = n - 2$(iii) $\gamma_{ds}(G) = n - 2$ and $\Delta(G) = n - 1$.

Case (i):$\gamma_{ds}(G) = n$ and $\Delta(G) = n - 3$
Since$\gamma_{ds}(G) = n$,G must be a star. But maximum degree of a star is $n - 1$ which is a contradiction.

Case (ii): $\gamma_{ds}(G) = n-1$ and $\Delta(G) = n-2$
Since $\gamma_{ds}(G) = n-1$, $DS = \{u_1, u_2, \dots u_{n-1}\}$ be the dom strong dominating set.Let $V - DS = \{v_1\}$ be the vertex other than the dom strong dominating set.

Subcase (i): Suppose $< DS >$ is disconnected
Suppose $|DS| = n$ where n is the number of vertices which are isolated. If n isolated vertices are adjacent to $v_1 \in V - DS$ then $G \cong K_{1,n}$ which is a contradiction.For $|DS| = 3$,Suppose$< DS >= K_2 \cup K_1$and $V - DS = \{v_1\}$.Let$u_1, u_2 \in K_2$ *and* $u_3 \in K_1$. If $d(u_1) = n-2, d(u_3) = n-3$then$G \cong P_4$.If $|DS| = 4$,then the possible cases are $< DS >= P_3 \cup K_1$(or) $K_3 \cup K_1$(or) $K_2 \cup K_2$ (or) $K_2 \cup \overline{K}_2$. Suppose $< DS >= P_3 \cup K_1$ and $V - DS = \{v_1\}$.Let $u_1, u_2, u_3 \in P_3$ where u_1,u_3 are pendant vertices ,$u_2$is a vertex of degree 2 *and* $u_4 \in K_1$. If $d(u_2) = n-2$ and $d(u_4) = n-4$ then G is a fork graph.If $d(u_2) = n-2$,$d(u_3) = n-3$and $d(u_4) = n-4$ then $G \cong K_3\ (P_2, P_2, 0)$. Suppose $< DS >= K_3 \cup K_1$and $V - DS = \{v_1\}$, since G is connected and any one vertex of K_3 is adjacent tov_1 then $G \cong K_3(P_3)$ for which the condition fails. Suppose $< DS >= K_2 \cup K_2$and $K_2 \cup \overline{K}_2$, we obtain graphs with$\gamma_{ds}\ (G) + \Delta(G) \neq 2n-3$.If$|DS| \geq 5$ and $< DS > = K_{1,n} \cup K_1$then $G\ \cong K_3\ (nP_2, P_2, 0)$ for n≥ 0.

Sub case(ii): Suppose $< DS >$ is connected.
If$|DS| \leq 3$, graphs with $\gamma_{ds}\ (G) + \Delta(G) \neq 2n-3$ is obtained. If $|DS| = 4$, then the possible cases are $< DS >= P_4$(or) K_4(or)$K_{1,3}$. Suppose $< DS >= P_4$and $V - DS = \{v_1\}$. Let $u_1, u_2, u_3, u_4 \in P_4$ where u_1, u_4 are the pendant vertices and u_2, u_3 are the vertices of degree 2. If $d(u_2) = d(u_3) = n-2$ then $G \cong K_3(P_2, P_2, 0)$. If $d(u_2) = n-2$ and $d(u_1) = n-3$ then$G \cong K_3(P_3)$ which is a contradiction. Suppose $< DS >= K_4$ and $K_{1,3}$and $V - DS = \{v_1\}$, graphs with $\gamma_{ds}\ (G) + \Delta(G) \neq 2n-3$ is obtained.

Case (iii): $\gamma_{ds}(G) = n-2$and $\Delta(G) = n-1$.
Since $\Delta(G) = n-1$, graph is complete. For complete graphs, $\gamma_{ds}(G) = 2$.Hence $G \cong K_4$.Since$\gamma_{ds}(G) = n-2$, $DS = \{u_1, u_2, \dots u_{n-2}\}$. Let$V - DS = \{v_1, v_2\}$be the vertices other than the dom strong dominating set.

Sub case (i): Suppose $< DS >$ is disconnectedand$< V - DS >= \overline{K}_2$ *or* K_2.
If $|DS| = 2$then two possible cases are $< DS >= \overline{K}_2$,$< V - DS >= \overline{K}_2$ and $< DS >= \overline{K}_2$,$< V - DS >= K_2$.Suppose $< DS >= \overline{K}_2$ and $< V - DS >= \overline{K}_2$ then$K_{2,2}$ exists, is a contradiction.If$|DS| \geq 3$,since $\Delta(G) = n-1$ graph does not exist.

Sub case (ii): Suppose $< DS >$ is connected and$< V - DS >= \overline{K}_2$ *or* K_2
If$|DS| = 1$, then $G \cong K_{1,2}$(or)K_3 which is a contradiction. If$|DS| = 2$, Suppose $< DS >= K_2$and $< V - DS >= \overline{K}_2$ then G is isomorphic to $K_4 - e$.Suppose $< DS >= K_2$ *and* $< V - DS >= K_2$then $G \cong K_4$.If$|DS| = 3$ then the possible casesare$< DS >= K_3(or)P_3$.Suppose $< DS >= P_3$,Let$u_1, u_2, u_3 \in P_3$ where u_1,u_3 are pendant vertices ,u_2 is a vertex of degree 2and$< V - DS >= \overline{K}\ \ _2$. If $d(u_2) = n-1$, $d(u_1) = d(u_3) = n-2$then$G \cong G_1$.If$(u_2) = n-1$,$d(u_1) = d(u_3) = n-3$ then $G \cong G_2$.If $d(u_2) = n-1$,$d(u_1) = n-2$, $d(u_3) = n-3$then $G \cong G_3$.If $d(u_1) = n-2, d(u_2) = n-1, d(u_3) = n-4$ then $G \cong G_5$.Suppose $< DS >= P_3$ and $< V - DS >= K\ \ _2$.If $d(u_2) = n-1$,$d(u_1) = n-2, d(u_3) = n-3$ then $G \cong G_4$. If the degree of dominating vertices is interchanged no new graphs exist.If $d(u_2) = n-1$, $d(u_1) = d(u_3) = n-3$ then $G \cong G_3$.If $d(u_1) = n-2, d(u_2) = n-1, d(u_3) = n-4$ then $G \cong G_6$.Suppose$|DS| \geq 4$, if $< DS > = K_{1,n}$and $< V - DS >= \overline{K}\ \ _2$ then G is isomorphic to $(K_4 - e)(nP_2)$. If$< DS > = K_{1,n}$ and $< V - DS >= K_2$ then G is isomorphic to $K_4(nP_2)$.

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